

## [I] Friction force

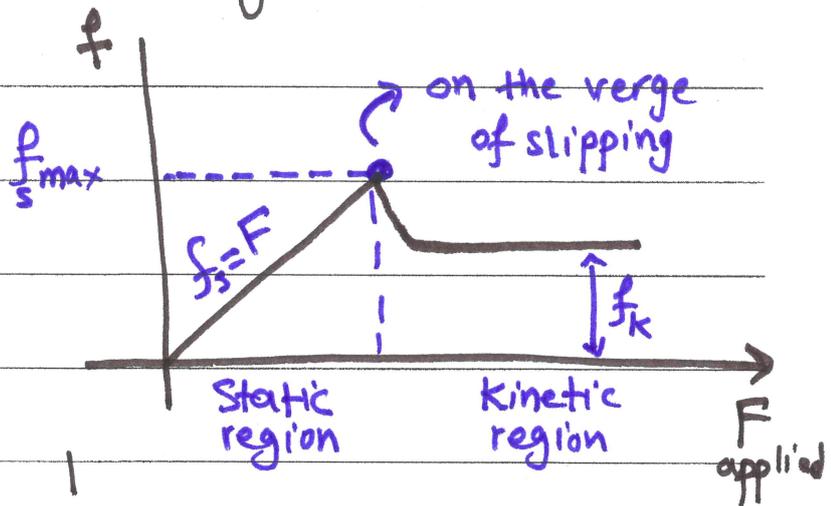
The basic characteristics of the force of friction are purely empirical, i.e., are based on experimental observations. Here are the findings we would obtain:



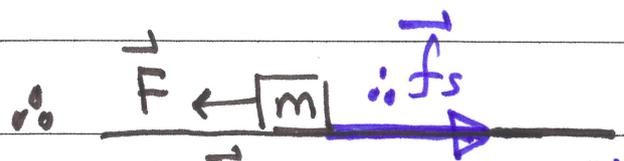
If you apply an external force  $\vec{F}$  to  $m$  and it remains stationary, then

according to Newton's 2<sup>nd</sup> law, the force that counteracts  $\vec{F}$  and keeps  $m$  at rest acts toward the right and is called static friction force  $\vec{f}_s$ .

The direction of  $\vec{f}_s$  is always opposite to that of  $\vec{F}$  relative to the surface of contact between



the object  $m$  and its supporting surface.



(ii) If  $\vec{F}$  is increased,  $\vec{f}_s$  also increases. When  $m$  is on the verge of slipping,  $\vec{f}_s$  has its maximum value  $f_{s\max}$  as shown above.

In other words, if  $m$  is at rest, it takes a certain threshold amount of external force  $\vec{F}$  to set it in motion.  $\Rightarrow F_{\text{thre.}} = f_{s \text{ max}}$ .

(iii) When  $F$  exceeds  $f_{s \text{ max}}$ ,  $m$  accelerates to the left. The friction force in motion is called kinetic friction force  $\vec{f}_k$ . The direction of  $\vec{f}_k$  is always opposite to the direction of motion of  $m$  relative to the surface it moves on:  $\vec{F} \leftarrow [m] \rightarrow \vec{f}_k$

(iv) Friction force depends on the roughness of the contact surface between  $m$  and the supporting surface. Thus both  $f_s$  and  $f_k$  are proportional to the magnitude of the normal force exerted by the rough contact surface. However,  $f_k$  is less than  $f_{s \text{ max}}$ . For the magnitude of  $f_s$ , we can write

$$f_s \leq \mu_s n, \quad f_{s \text{ max}} = \mu_s n \quad \text{--- (1)}$$

$\mu_s$  is the coefficient of static friction. Likewise,  $f_k = \mu_k n$  --- (2), where  $\mu_k$  is the coefficient of kinetic friction.

$$\Rightarrow f_{s \text{ max}} > f_k \Rightarrow \mu_s > \mu_k \quad \text{--- (3)}$$

Some typical coefficients of static and kinetic friction are shown in Table 4.2 in your text.

The case where  $\mu = 0$  corresponds to a frictionless approximation (idealized smooth surface).

(v) The net force on  $m$  is  $[F - f_k]$  in the  $x$ -direction, produces an acceleration to the left. If  $F = f_k$ ,  $a = 0$  and  $m$  can move to the left with constant speed. If  $\vec{F}$  is removed from the moving object  $m$ , the  $\vec{f}_k$  acting to the right provides an acceleration and eventually brings  $m$  to rest, according to Newton's 2<sup>nd</sup> law.

(vi). Equations (1) and (2) are not vector equations! Because  $f$  and  $n$  are perpendicular to each other, the vectors can NOT be related by a multiplicative constant. They are relationships between the magnitudes of the vectors representing the friction and normal forces.

## [II] Applications of the friction force.

Knowing about static and kinetic friction allows us to approximate real-world situations and come to meaningful conclusions.

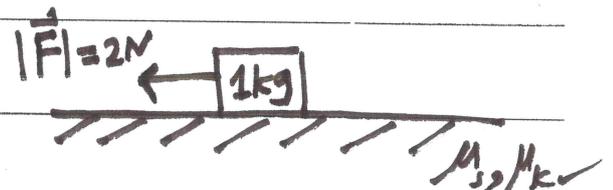
[I] A typical exam problem; you better watch out...!

A 1 kg block rests on a rough horizontal surface.

Then, a pull force of magnitude 2 N is applied to the block as shown. Knowing that  $\mu_s = 0.3$  and  $\mu_k = 0.1$ , what is the friction force,  $\vec{f}$ , acting on the block? Take  $g = 10 \text{ m/s}^2$ ?

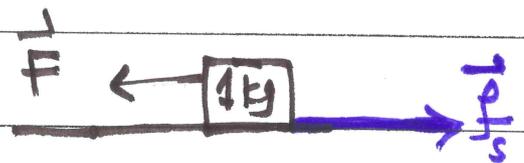
Eq. (1) reads:  $f_{s \max} = \mu_s n$

$$f_{s \max} = \mu_s mg = 3 \text{ N.}$$



Thus  $|F|$  is less than the threshold,  $f_{s \max}$ , needed to set the block into motion. In that case, according to Newton's 2<sup>nd</sup> law,  $\vec{f}$  is  $\vec{f}_s$  not  $\vec{f}_{s \max}$  neither  $\vec{f}_k$

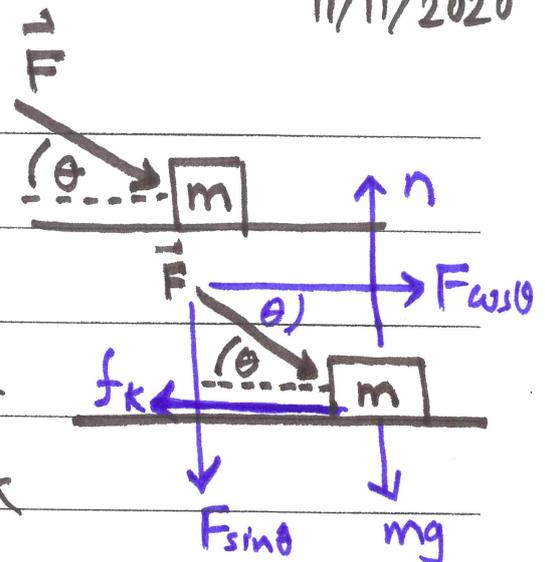
$\therefore \vec{f} = 2 \text{ N}$  to the right



\* Rationally, argue why the answers  $|\vec{f}| = 3 \text{ N}$  and  $|\vec{f}| = 1 \text{ N}$  are incorrect!

**2** Past exam:

As shown, the block  $m$  is pushed across a horizontal surface by the force  $\vec{F}$ . Knowing that the coefficient of kinetic friction between the block and the rough surface is 0.30, what is the magnitude of the acceleration of the block?



Given:  $m = 3 \text{ kg}$ ,  $|\vec{F}| = 20 \text{ N}$ ,  $\theta = 30^\circ$ .

$\Rightarrow$  Newton's 2<sup>nd</sup> law:  $F \cos \theta - f_k = ma$ , (recall item (v))

$\Rightarrow$  Eq (2):  $f_k = \mu_k n \Rightarrow n \neq mg !!$ ,  $n = mg + F \sin \theta$

$$\therefore a = \frac{1}{m} [F \cos \theta - \mu_k (mg + F \sin \theta)] \quad (*)$$

$\Rightarrow$  Inserting the numbers given in the problem statement, we obtain  $a = 1.79 \text{ m/s}^2$ .

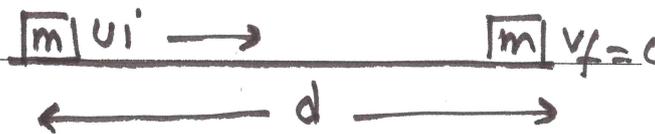
Validation: What does eq (\*) reduce to if the angle  $\theta$  goes to zero? Does this algebraic expression match your intuition?

Compare the results of ex 4.18 and ex 4.11.

Do ex 4.17.

**3** Past exam: A mass  $m = 0.5 \text{ kg}$  on a frozen pond is given an initial speed of  $20 \text{ m/s}$ . If the mass always remains on the ice and slides  $115 \text{ m}$  before coming to rest, determine the coefficient of kinetic friction between the mass and ice.

Recall item (v) page 3:  $f_k$  slows the mass, which eventually comes to rest due to that force.

Imagine the mass  slides to the right, then

$f_k$  acts to the left

$$\therefore -f_k = ma, \quad n = mg \Rightarrow -\mu_k mg = ma$$

$$\therefore a = -\mu_k g \quad \text{--- (1)}$$

Because the velocity of the mass is to the right, the mass is slowing down: the -ve sign means that  $a$  is to the left.  $\Rightarrow a$  is constant,  $\Rightarrow$

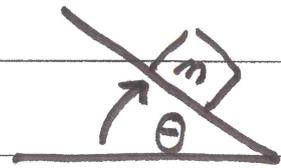
$$v_f^2 = v_i^2 + 2\vec{a}\vec{d} \Rightarrow 0 = v_i^2 + 2(-\mu_k g)(+d)$$

$$\therefore \mu_k = \frac{v_i^2}{2gd} \quad \text{--- (2)} = 0.177$$

$\frac{6}{16}$

Notice that  $\mu_k$  ( $\frac{2}{1}$  over  $\frac{2}{2}$ ) is dimensionless, as it should be, and it has a low value, consistent with an object sliding on ice.

#### 4] Experimental determination of $\mu_s$ .

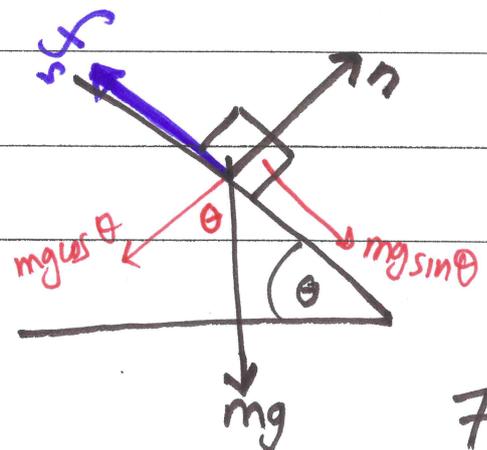


Suppose a block  $m$  is placed on a rough surface inclined relative to the horizontal as shown. The incline angle  $\theta$  is increased until the block starts to move. Determine  $\mu_s$  by measuring the critical angle  $\theta_c$  at which the block is just on the verge of slipping.

⇒ Notice how this problem differs from problem 8, lecture 5. When there is no friction on an incline, any angle of the incline will cause a stationary block to begin moving. When there is friction, however, there is no motion for angles less than the critical angle  $\theta_c$ .  $\theta_c$  is the angle at which the block is just ready to begin to move but is not moving.

$$\circledast \quad mg \sin \theta = f_s \quad (1)$$

$$n = mg \cos \theta \quad (2)$$



When the incline angle is increased until it equals  $\theta_c$ , the force  $f_s$  has reached its maximum value  $f_{s, \max}$ , i.e., as  $\theta = \theta_c \Rightarrow f_s = f_{s, \max}$  — (3).

$$\Rightarrow \text{Eq (1) reads: } mg \sin \theta_c = f_{s, \max} = \mu_s n \quad (*)$$

$\Rightarrow$  Substitute  $n = mg \cos \theta$  from eq (2) into (\*):

$$mg \sin \theta_c = \mu_s mg \cos \theta_c \Rightarrow \boxed{\mu_s = \tan \theta_c} \quad (4)$$

Thus, we have shown that  $\mu_s$  is related only to  $\theta_c$ .

■ For example, if the block just slips at  $\theta_c = 20^\circ$ , then  $\mu_s = 0.364$ .

■ Once the block starts to slip at  $\theta \geq \theta_c$ , it accelerates down the incline and the force of friction is  $f_k = \mu_k n$ .

■ Do problem 4.38 and ex 4.21.

■ Past exam: For the same incline, take

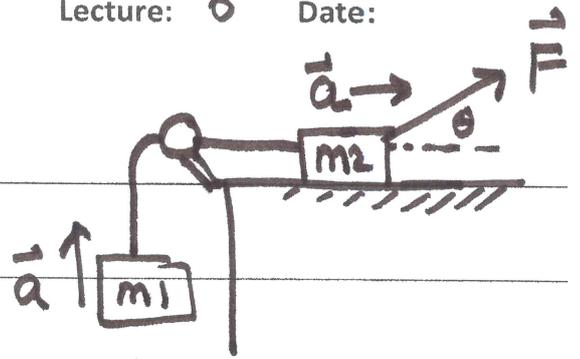
$m = 13 \text{ kg}$ ,  $\theta$  of the incline  $= 14^\circ$ , the block moves down the incline with initial speed of  $1.4 \text{ m/s}$ ,  $\mu_k = 0.38$ .

What is the acceleration of the block?

$$a = g [\sin \theta - \mu_k \cos \theta] = -1.24 \text{ m/s}^2$$

■ Quiz: If the block moves down the incline with constant speed, then  $a = g [\sin \theta - \mu_k \cos \theta] = 0 \Rightarrow \boxed{\mu_k = \tan \theta}$ .

[5] A block of mass  $m_2$  on a rough, horizontal surface is connected to a block of mass  $m_1$  by a cord over a pulley as shown.



A force of magnitude  $F$  at an angle  $\theta$  with the horizontal is applied to  $m_2$ , and the block slides to the right. The coefficient of kinetic friction between  $m_2$  and surface is  $\mu_k$ . Determine the magnitude of the acceleration of the two blocks.

$$F \cos \theta - f_k - T = m_2 a \quad \text{--- (1)}$$

$$T - m_1 g = m_1 a \quad \text{--- (2)}$$

$$n = m_2 g - F \sin \theta \quad \text{--- (3)}$$

$$\Rightarrow f_k = \mu_k n = \mu_k (m_2 g - F \sin \theta)$$

Substitute  $f_k$  and the value

of  $T$  from eq (2) into eq (1) and solve for  $a$ :

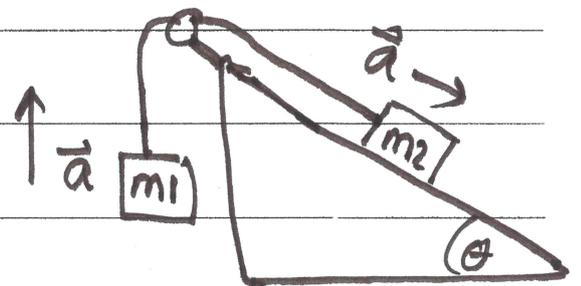
$$a = \frac{F (\cos \theta + \mu_k \sin \theta) - (m_1 + \mu_k m_2) g}{(m_1 + m_2)} \quad \text{--- (4)}$$

\* Let's elaborate on problem [5]:

① What does eq (4) reduce to if the force  $\vec{F}$  is removed and the surface becomes frictionless? Call this expression eq (5). Does this algebraic expression match your intuition about the physical situation in this case?

② Now solve for the acceleration of the two blocks for the following problem:

The two blocks  $m_1, m_2$  are attached by a cord that passes over a pulley. Block  $m_2$



lies on a frictionless incline of angle  $\theta$ . Find the magnitude of the acceleration of the two blocks.

Call this expression eq (6).

③ Now let angle  $\theta$  go to zero in eq (6). How does the resulting equation compare with your equation (5) here in problem [5]?

④ Problem (2) above becomes the Atwood machine of problem [7] lecture 5, by letting  $\theta \rightarrow 90^\circ$  in eq (6) above. Verify that eq (6) above reduces to the result of page 7 lecture 5.

[6] Problem 4.44, Problem 4.45 and Example 4.20:

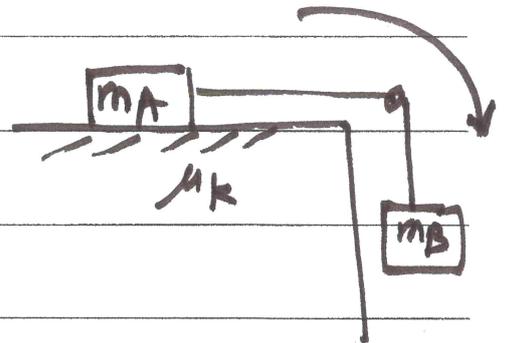
(i) Read example 4.20:

Solve for the acceleration  $a$ :

$$m_B g - T = m_B a$$

$$T - \mu_k m_A g = m_A a$$

$$\Rightarrow a = \left( \frac{m_B - \mu_k m_A}{m_A + m_B} \right) g \quad \text{--- [1]}$$



Compare your result with that of problem [7] lecture 5.

If  $m_B \gg m_A \Rightarrow a = g \Rightarrow m_B$  is in free fall ✓.

(ii) Problem 4.44: For the same system: what is  $m_A$  needed to prevent any motion from occurring?

$\Rightarrow$  We need  $\mu_s$  not  $\mu_k$  and let  $a$  of [1] = 0

$$\Rightarrow \text{then } m_B = \mu_s m_A \Rightarrow m_A = \frac{m_B}{\mu_s} = \frac{2}{0.3} \text{ kg.}$$

(iii) Problem 4.45: For the same system:

(a) What is  $m_A$  needed to keep the system from starting to move?  $\Rightarrow$  let  $a$  of [1] = 0  $\rightarrow m_A = \frac{m_B}{\mu_s} = \frac{2}{0.4} \text{ kg.}$

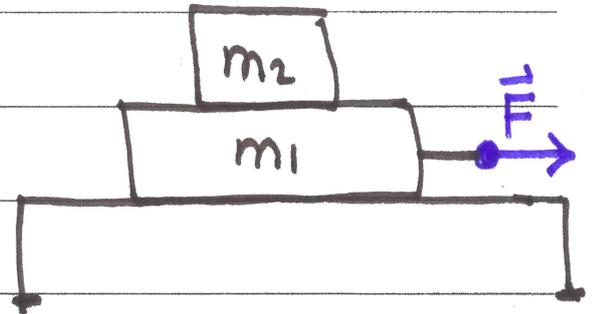
(b) What is  $m_A$  needed to keep the system moving at constant speed?  $\Rightarrow$  let  $a$  of [1] = 0

$$\rightarrow m_A = \frac{m_B}{\mu_k} = \frac{2}{0.2} \text{ kg}$$

[7] A "challenge" problem: Two stacked blocks:

Consider two stacked blocks on a table as shown.

The upper block has a mass  $m_2$ , and the lower block has a mass  $m_1$ .



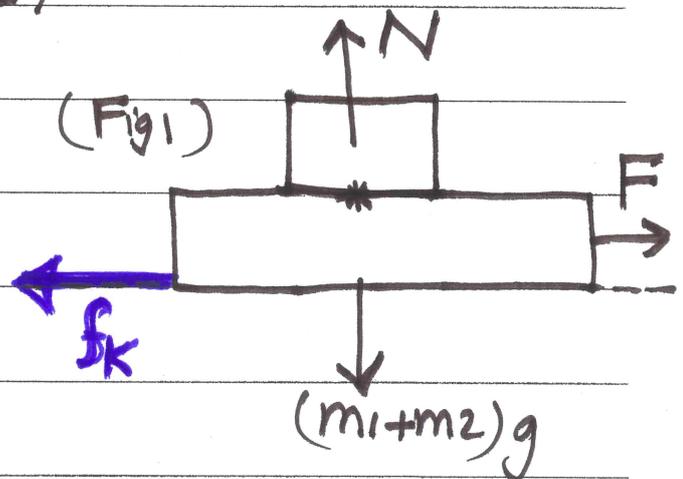
The coefficient of kinetic friction between  $m_1$  and the table is  $\mu_k$ . The coefficient of static friction between the blocks is  $\mu_s$ . A string is attached to  $m_1$ , and an external force  $\vec{F}$  is applied horizontally, pulling on the string as shown.

■ What is the maximum force that can be applied to the string without having the upper block slide off?

(1) Notice that as long as the force of static friction between the two blocks is not overcome, the two blocks will travel together. Thus, if we pull gently on  $m_1$ ,  $m_2$  will stay in place on top of it, and

the two blocks will slide as one. However, if we pull hard on  $m_1$ , the force of static friction between the blocks will not be sufficient to keep  $m_2$  in place and it will begin to slide off  $m_1$ .

(2) Let us start by treating the two blocks as one system moving together. The forces acting in this case are the external force  $F$  pulling on the string, the force of kinetic friction  $f_k$  between  $m_1$  and the surface on which the blocks are sliding, the weight  $m_1g$  of the lower block, the weight  $m_2g$  of the upper block and the normal force exerted by the table (as shown in Figure 1):



■ Please read page

2 of this lecture:

$f_k$  is always opposite to the direction of motion

of the object relative to the surface it moves on (table).

∴  $f_k$  is indeed to the left.

$$\Rightarrow F - f_k = (m_1 + m_2) a \quad \text{--- (1)}$$

$$N = (m_1 + m_2) g \quad \text{--- (2)}$$

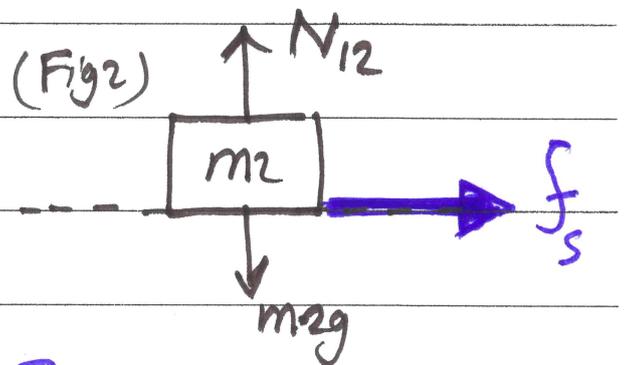
$$f_k = \mu_k N = \mu_k (m_1 + m_2) g \quad \text{--- (3)}$$

$$\therefore F = (m_1 + m_2) (a + \mu_k g) \quad \text{--- (4)}$$

Eq (4) shows that the maximum force is achieved by determining the maximum acceleration  $a$ .

(3) Thus, we need to analyze the forces acting on the upper block  $m_2$ . The forces for  $m_2$  are the normal force exerted by the lower block  $N_{12}$ , the weight  $m_2 g$ , and the force of static friction  $f_s$  (as shown in Figure 2):

■ Please "ponder" the direction of  $f_s$ : In page 1 of this lecture we said that the direction of  $f_s$  is always opposite to that of  $F$  relative to the surface of contact between  $m_2$  and its supporting surface. This relative surface is **NOT** the surface of the table, but rather the surface of  $m_1$ .



$$\Rightarrow f_s = m_2 a \quad \text{--- (5)}$$

$$N_{12} = m_2 g \quad \text{--- (6)}$$

The maximum value of  $f_s$  between the upper and lower blocks is given by:  $f_{s \max} = \mu_s N_{12} = \mu_s m_2 g$ .

Thus, the maximum acceleration of that the upper block can have without sliding is given by eq (5)

$$a_{\max} = \frac{f_{s \max}}{m_2} = \frac{\mu_s m_2 g}{m_2} \Rightarrow a_{\max} = \mu_s g \quad \text{--- (7)}$$

(4) This maximum acceleration for  $m_2$  is also the maximum acceleration for both blocks together. We can now relate the maximum acceleration to the maximum force,  $F_{\max}$ , that can be exerted without the upper block sliding off using eq (7) and eq (4):

$$\therefore F_{\max} = (m_1 + m_2) (\mu_s g + \mu_k g)$$

$$\therefore \underline{F_{\max} = (\mu_k + \mu_s) (m_1 + m_2) g} \quad \text{--- (8)}$$

(5) Validation: If there were no friction between the lower block  $m_1$  and the surface of the table, the force required to accelerate both blocks without the upper one ~~from~~ sliding off would be:

recall eq (1)  $\Rightarrow F = (m_1 + m_2)a$

recall eq (7)  $a_{\max} = \mu_s g$

$\therefore F_{\max} = (m_1 + m_2)\mu_s g$  \_\_\_\_\_ (9)

However, this result could be obtained

directly by letting  $\mu_k = 0$  in eq (8) ✓.

Thus, our answer (8) for the maximum force seems reasonable because it is higher than the force (9) calculated when there is no friction!